1. (a) In odometry, the rotation of the two wheels is determined using wheel encoders. Wheel encoders consist of a disc of glass or plastic, with shaded regions that interrupt a light beam. By counting the number of interruptions, the rotation of the wheel can be deduced. Using two detectors, placed a quarter of a cycle out of phase with each other, the direction of rotation can be deduced as well. Given the wheel rotation speeds V_{L} and V_{R}, the position and heading of the robot can be computed, using Eqs. (2.7)-(2.9) in Chapter 2.

   (b) Localization can be carried out by means of scan matching. In this process, the rays of a laser range finder (LRF) are matched against the rays of a virtual LRF giving simulated readings using a map (in the form of a sequence of lines). The virtual LRF is placed at a variety of different positions, starting with the position corresponding to the current pose obtained from odometry. If this pose generates an error (in the scan match) above a certain threshold, as search procedure is carried out in order to find a pose giving a better match. Once such a pose has been found, the odometric estimate is recalibrated accordingly.

   (c) The St. Petersburg paradox introduces a lottery such that a person should be willing to pay an infinite amount of money to participate, given that the expected amount of money gained from the lottery was the only factor determining a person's willingness to participate. In the lottery introduced in the St. Petersburg paradox, the expected payoff P is infinite:

$$P = -r + \sum_{k=1}^{\infty} p_k c_k = -r + \sum \left(\frac{1}{2}\right)^k 2^k = -r + \sum_{k=1}^{\infty} 1 \tag{1}$$

   where r is the cost of entering the lottery, p_k is the probability of outcome c_k (in which the participant wins 2^k currency units).

   Clearly, few people would be willing to pay all their money to participate in the lottery, and the paradox can be resolved by noting (as did Bernoulli) that it is the *perception* of the expected payoff, rather than the payoff itself, that determines a person's willingness to participate in the lottery. Thus, in this case, the subjective value P_s of the payoff would be

$$P_s = \sum_{k=1}^{\infty} (\ln(W - r + 2^k) - \ln W) 2^{-k}, \tag{2}$$

   where W is the participant's wealth before entering the lottery. P_s is finite, thus resolving the paradox.

   (d) DC motors consist of a stator, rotor, commutator, and (in some cases) brushes. The stator provides the magnetic field. The functionality is described on pp. 12-13 in Chapter 2.

   (e) The generated torque is derived on pp. 13-14 in Chapter 2. Only the electrical equations need be considered. Letting V denote the applied voltage, and ω the angular speed of the motor shaft, the electrical equation takes the form

$$V = L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + V_{\mathrm{EMF}}, \tag{3}$$

where i is the current flowing through the circuit, L is the inductance of the motor, R its resistance, and V_{EMF} the back EMF, given by

$$V_{\mathrm{EMF}} = c_e \omega, \tag{4}$$

where c_e is the electrical constant of the motor. τ_g is directly proportional to the current, i.e.

$$\tau_g = c_t i, \tag{5}$$

where c_t is the torque constant of the motor. Putting together the expressions above, one obtains, neglecting the transient $\mathrm{d}i/\mathrm{d}t$:

$$\tau_g = \frac{c_t}{R} V - \frac{c_e c_t}{R} \omega. \tag{6}$$

2. (a) The complete potential field can be written as

$$\Phi(x,y) = \alpha_g \mathrm{e}^{-\left(\frac{x-x_g}{\beta_g}\right)^2 - \left(\frac{y-y_g}{\gamma_g}\right)^2} + \alpha_o \mathrm{e}^{-\left(\frac{x-x_o}{\beta_o}\right)^2 - \left(\frac{y-y_o}{\gamma_o}\right)^2} \tag{7}$$

The movement direction suggested by the potential field is given as

$$\hat{\mathbf{r}} = -\frac{\nabla\Phi}{|\nabla\Phi|} = -\frac{\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial x}\right)}{|\nabla\Phi|}. \tag{8}$$

The two partial derivatives are given by

$$\frac{\partial\Phi}{\partial x} = -2(x - x_g)\frac{\alpha_g}{\beta_g^2}\mathrm{e}^{-\left(\frac{x-x_g}{\beta_g}\right)^2 - \left(\frac{y-y_g}{\gamma_g}\right)^2} - 2(x - x_o)\frac{\alpha_o}{\beta_o^2}\mathrm{e}^{-\left(\frac{x-x_o}{\beta_o}\right)^2 - \left(\frac{y-y_o}{\gamma_o}\right)^2} \tag{9}$$

and

$$\frac{\partial\Phi}{\partial y} = -2(y - y_g)\frac{\alpha_g}{\gamma_g^2}\mathrm{e}^{-\left(\frac{x-x_g}{\beta_g}\right)^2 - \left(\frac{y-y_g}{\gamma_g}\right)^2} - 2(y - y_o)\frac{\alpha_o}{\gamma_o^2}\mathrm{e}^{-\left(\frac{x-x_o}{\beta_o}\right)^2 - \left(\frac{y-y_o}{\gamma_o}\right)^2}, \tag{10}$$

from which the numerator can be obtained. The denominator is then given as the square root of the sum of squares of the components in the numerator.

(b) Inserting numerical values, one obtains the direction vector as $(0.9601, -0.2795)$.

(c) Moving one length unit in the direction obtained in (b), one arrives at the point $(3.9601, -1.2795)$. At this point, the direction vector is $(0.9589, 0.2836)$.

3. (a) The free-body diagram can be seen in Fig. 2.3 in Chapter 2. With the notation in Fig. 2.3, the dynamic equations are

$$m\dot{v}_L = F_L - \rho_L, \tag{11}$$

$$m\dot{v}_R = F_R - \rho_R, \tag{12}$$

$$\overline{I}_{\mathrm{w}}\ddot{\phi}_L = \tau_L - F_L r, \tag{13}$$

$$\overline{I}_{\mathrm{w}}\ddot{\phi}_R = \tau_R - F_R r, \tag{14}$$

$$M\dot{V} = \rho_L + \rho_R \tag{15}$$

and

$$\overline{I}\ddot{\varphi} = \left(-\rho_L + \rho_R\right) R, \tag{16}$$

(b) The kinematic equations are

$$v_L = r\dot{\phi}_L \tag{17}$$

and

$$v_R = r\dot{\phi}_R. \tag{18}$$

$$V = \frac{v_L + v_R}{2}, \tag{19}$$

$$\dot{\varphi} = -\frac{v_L - v_R}{2R}. \tag{20}$$

(c) The derivation is given in Chapter 2, pp. 24-27. Neglecting damping and losses, the final equations are

$$M\dot{V} = \frac{1}{r}\left(\tau_{\mathrm{L}} + \tau_{\mathrm{R}}\right) \tag{21}$$

and

$$\overline{I}\ddot{\varphi} = \frac{R}{r}\left(-\tau_{\mathrm{L}} + \tau_{\mathrm{R}}\right), \tag{22}$$

4. (a) $u(c_1) = u(pc_0 + (1-p)c_{100}) = (\text{Eq } (7.4)) =$
$= pu(c_0) + (1-p)u(c_{100}) = 0.5 \times 0 + 0.5 \times 100 = 50.$

(b) Person A is prefers to accept 30 certain dollars, rather than gambling for the opportunity of receiving 100 dollars, whereas person B would prefer the gamble up to a point where he would receive 60 (certain) dollars. Thus A is risk-averse, and B is risk-seeking.

5. (a) The description of the localization method should include (i) definition of the virtual LRF, (ii) the equation for scan matching, (iii) the search procedure: Starting at the current odometric estimate, and then proceeding to search in a cube around that position, if the matching error exceeds a certain threshold.

(b) Basic geometric considerations give (for the actual LRF): $r_1 = 2.25$, $r_2 = -1$ (no detection), $r_3 = 2.0$, $r_4 = -1$ (no detection) and $r_5 = 2.25$.

(c) For the virtual LRF, one obtains $r_1 = 2.0$, $r_2 = 0$ (no detection), $r_3 = 1.5$, $r_4 = 2.12$ and $r_5 = 2.5$. Thus, the scan match error becomes

$$\sqrt{\frac{1}{3}\left(\left(1 - \frac{2}{2.25}\right)^2 + \left(1 - \frac{1.5}{2}\right)^2 + \left(1 - \frac{2.5}{2.25}\right)^2\right)} = 0.1705. \tag{23}$$